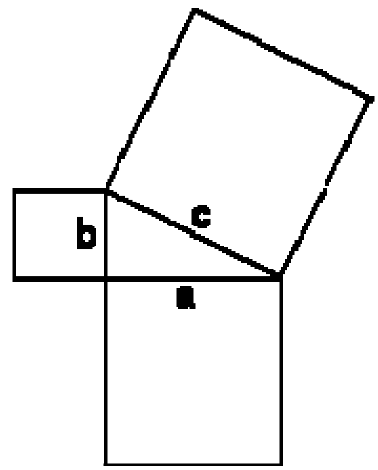
Topic: Trigonometry

Pre-requisite: Knowledge of Pythagoras Theorem

Recap of Pythagoras Theorem



Pythagoras' Theorem: the sum of the areas of two small squares equals the area of the big one.

In algebraic terms, $a^2 + b^2 = c^2$ where c is the *hypotenuse* while a and b are the lengths of the other two sides of a right-angled triangle.

What is Trigonometry?

Trigonometry is a branch of mathematics concerned with specific functions of *angles* and their application to calculations.

Trigonometry relates to the field of geometry. For instance, given one side of a triangle and two angles in the triangle, then the other two sides and the remaining angle can be determined. Trigonometry includes the methods for computing those other two sides. It allows indirect measurement of angles and length of objects in practical situations. By knowing basic Trigonometry concepts, we can calculate the height of a tree in our school compound or the height of the Tan Kah Kee statue, without actually measuring the height physically.

Why learn Trigonometry?

Historically, it was developed for astronomy and geography, trigonometric tables were created over two thousand years ago for computations in astronomy.

The stars were thought to be fixed on a crystal sphere of great size, and that model was perfect for practical purposes. The kind of trigonometry needed to understand positions on a sphere is called ***spherical trigonometry***. Spherical trigonometry is rarely taught now since its job has been taken over by linear algebra.

As the earth is also a sphere, trigonometry is used in geography and in navigation. Ptolemy (100-178, Greek astronomer) used trigonometry in his *Geography* and used trigonometric tables in his works.

Columbus carried a copy of **Regiomontanus'** ***Ephemerides Astronomicae*** on his trips to the New World and used it to his advantage.

Although trigonometry was first applied to spheres, it has had greater application to planes. Surveyors have used trigonometry for centuries. Engineers, both military engineers and otherwise, have used trigonometry nearly as long.

Physics lays heavy demands on trigonometry. Optics and statics are two early fields of physics that use trigonometry, but all branches of physics use trigonometry since it aids in understanding space. Related fields such as physical chemistry naturally use trigonometry.

Within mathematics, trigonometry is used primarily in calculus, which is perhaps its greatest application, linear algebra, and statistics.

Since these fields are used throughout the natural and social sciences, trigonometry is a very useful subject to know.

Definition

There are six **functions of an angle** commonly used in trigonometry. Their names and abbreviations are:

sine (*sin*)	**cosine (*cos*)**	**tangent (*tan*)**
cosecant (*cosec*)	**secant (*sec*)**	**cotangent (*cot*)**

Note: They are all ratios and so have no units.

You will learn about the Sine, Cosine and Tangent Ratios in Sec 2. Cotangent, Secant and Cosecant will be taught in Sec 3.

In a full circle, an angle spans 360°. In trigonometry, we usually use this symbol "θ" (pronounced as *theta*) to denote angles.

$0^\circ < \theta < 90^\circ$ → Acute Angles

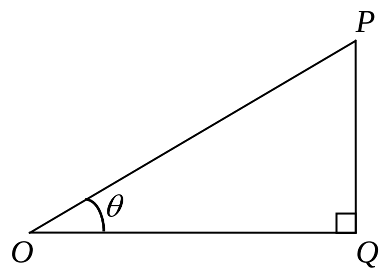
$90^0 < \theta < 180^\circ$ → Obtuse Angles

$\theta > 180^\circ$ → Reflex Angles

Note: For Sec 2, we are only concerned with Trigo Ratios of Acute Angles.

Trigonometric Ratios of Acute Angles

Just like Pythagoras Theorem, Trigo Ratios can only be applied in Right-Angled Triangles.



The 3 Trigo Ratios are defined as follows in a right-angled triangle:

$$\sin\theta = \frac{\text{length of opposite side}}{\text{length of hypotenuse}} = \frac{PQ}{OP}$$

$$\cos\theta = \frac{\text{length of adjacent side}}{\text{length of hypotenuse}} = \frac{OQ}{OP}$$

$$\tan\theta = \frac{\text{length of opposite side}}{\text{length of adjacent side}} = \frac{PQ}{OQ}$$

Note: It is important to identify the opposite and adjacent sides relative to the angle θ.

Sine ratio can be used for computing an acute angle when the *opposite* and *hypotenuse* are known.

Cosine ratio can be used for computing an acute angle when the *adjacent* and *hypotenuse* are known.

Tangent ratio can be used for computing an acute angle when the *opposite* and *adjacent* are known.

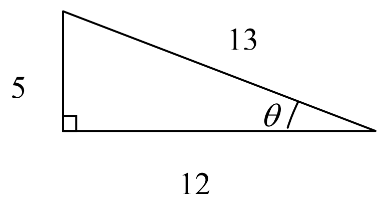
Since **s**in θ = $\frac{\textbf{O}\text{pposite}}{\textbf{H}\text{ypotenuse}}$, **c**os θ = $\frac{\textbf{A}\text{djacent}}{\textbf{H}\text{ypotenuse}}$ and **t**an θ = $\frac{\textbf{O}\text{pposite}}{\textbf{A}\text{djacent}}$

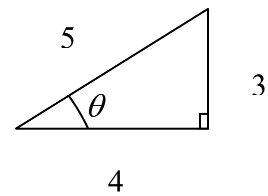
it is a good idea to memorise these three ratios as:

TOA CAH SOH

Exercise 1

Using the definition of basic trigo ratios, fill in the following table:

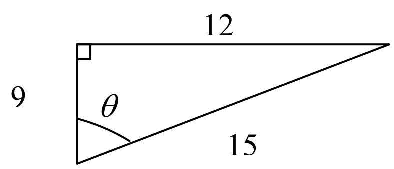


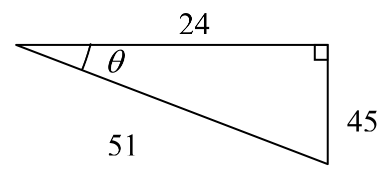


$\sin\theta$ = ___________ $\sin\theta$ = ___________

$\cos\ \theta$ = __________ $\cos\ \theta$ = __________

$\tan\ \theta$ = __________ $\tan\ \theta$ = __________





$\sin\theta$ = ___________ $\sin\theta$ = ___________

$\cos\ \theta$ = __________ $\cos\ \theta$ = __________

$\tan\ \theta$ = __________ $\tan\ \theta$ = __________

Exercise 2

Find, by sketching a right triangle, the other two trigonometric ratios of angles *A*, *B* and *C*:

(a) $\sin A = \frac{3}{5}$ (b) $\cos B = \frac{8}{17}$ (c) $\tan C = \frac{21}{20}$

<u>Exercise 3</u>

Using your calculator, find the following trigonometric ratios to 3 significant figures.

(a) sin 60° =　　　　　　　　(b) cos 60° =

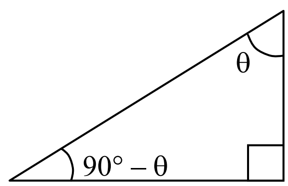
(c) tan 45° =　　　　　　　　(d) cos 30° =

(e) sin 0.5° =　　　　　　　　(f) tan 23° =

From the above, what can you gather about the ratios of complementary angles?

Note: Complementary angles add up to 90°.

Since (90° – θ) and θ are complementary, $\boxed{\sin(90° - \theta) = \cos\theta.}$



<u>Exercise 4</u>

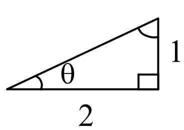
True or False? Use actual values and check using your calculator.

(a) $\sin(\theta \div 2) = \sin\theta \div 2$

(b) $\cos 2\theta = 2 \times \cos\theta$

(c) $\tan(A + B) = \tan A + \tan B$

Finding Angles from the Ratios



For the triangle on the left, we can use a protractor to measure the angle of θ. However, it would be troublesome of do so for all angles. By knowing the trigo ratio for the angle, we can easily calculate its value.

First, write out the trigo ratio using the sides given. Then, we will take the *inverse* of the ratio to find the angle.

<u>Working:</u>

Now, go back to Exercise 1 and find the angle θ.

Exercise 5

Solve the following for $0° \le \theta \le 90°$. Give your answers for angles correct to 1 decimal place.

(a) $\sin\theta = 0.4537$

(b) $\cos\theta = 0.3625$

(c) $\tan\theta = 4.393$

(d) $\sin\theta = 0.8888$

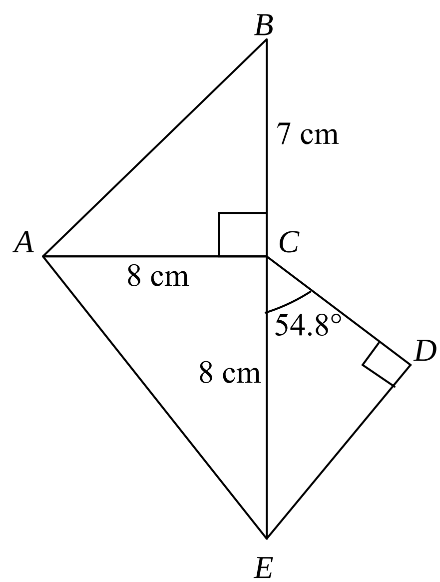
(e) $\cos\theta = 0.9999$

(f) $\tan\theta = 0.5177$

Further Examples

1 In the diagram, *BCE* is a straight line, $\angle ECD = 54.8°$ and $\angle CDE = \angle ACB = 90°$. $BC = 7$ cm and $AC = CE = 8$ cm. Calculate

(a) $\angle CED$,
(b) $\angle DCB$,
(c) the length of *AE*,
(d) $\angle BAC$,
(e) the length of *ED*,



2 A 16 m ladder is leaning against a house. It touches the bottom of a window that is 12 m above the ground. What is the measure of the angle that the ladder forms with the ground?

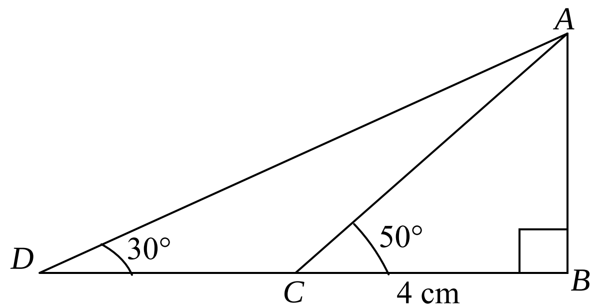
Exercise 6

Q1 In the diagram, $\angle ADC = 30°$, $\angle ACB = 50°$, $\angle ABD = 90°$ and $BC = 4$ cm. Calculate

(a) $\angle DAC$,

(b) AB,

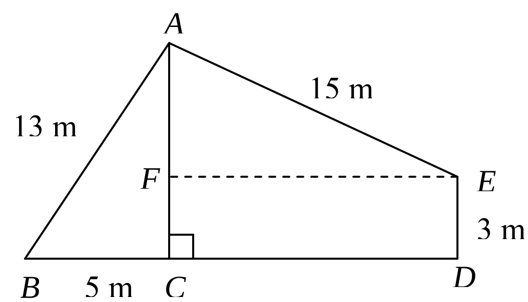
(c) AD.



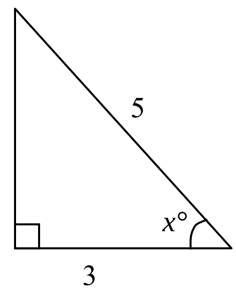
Q2 In the diagram given below, find the length of

(a) AC,

(b) EF.



Q3 Given the triangle below, find the value of 3 sin x − cos x.

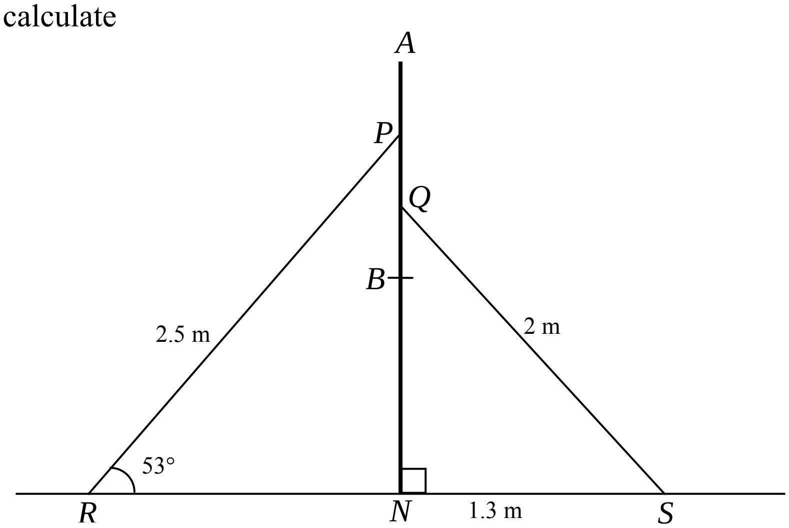


Q4 A pole AN is held in place by two supports PR and QS as shown in the diagram. R and S are points where the supports touch the horizontal ground. $PR = 2.5$ m, $QS = 2m$, $NS = 1.3$ m, $P\hat{R}N = 53°$ and $AP = PQ = QB$, calculate

(a) AN,

(b) BN.

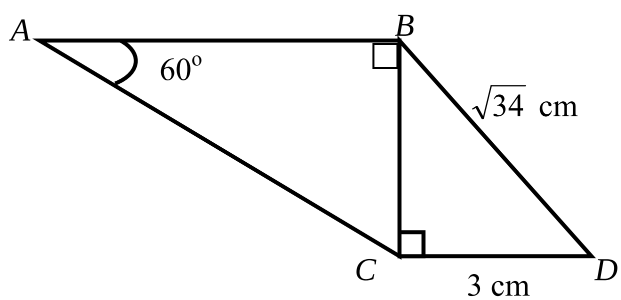
.



Q5 Calculate the length of

(a) BC

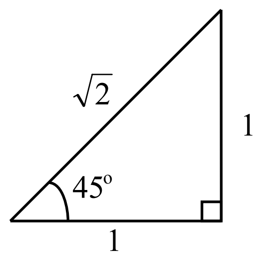
(b) AB

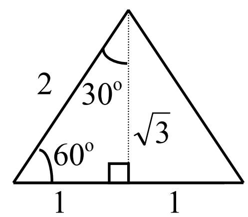


Trigonometric Ratios of Special Angles

In trigonometry, we refer to special angles as 30°, 45° and 60°.

We see that





$$\cos 45^{\circ} = \frac{1}{\sqrt{2}} \qquad \cos 60^{\circ} = \frac{1}{2} \qquad \cos 30^{\circ} = \frac{\sqrt{3}}{2}$$

$$\sin 45^{\circ} = \frac{1}{\sqrt{2}} \qquad \sin 60^{\circ} = \frac{\sqrt{3}}{2} \qquad \sin 30^{\circ} = \frac{1}{2}$$

$$\tan 45^{\circ} = 1 \qquad \tan 60^{\circ} = \sqrt{3} \qquad \tan 30^{\circ} = \frac{1}{\sqrt{3}}$$

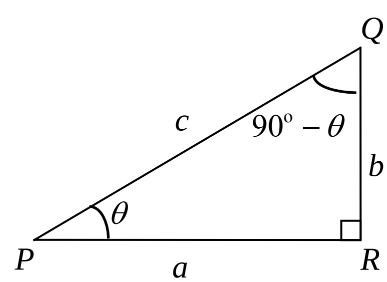
Trigonometric Ratios of Complementary Angles

We see that

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{a}{c} \qquad \cos(90^{\circ} - \theta) = \frac{b}{c}$$

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{b}{c} \qquad \text{and} \qquad \sin(90^{\circ} - \theta) = \frac{a}{c}$$

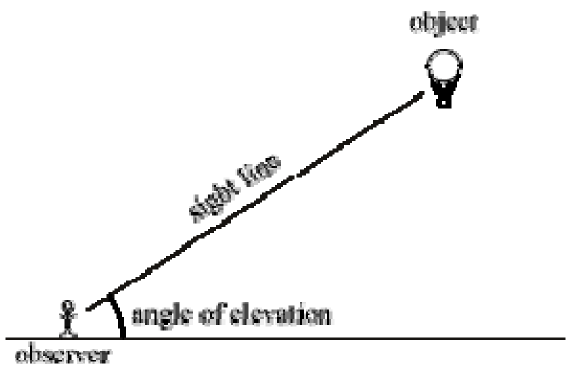
$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{b}{a} \qquad \tan(90^{\circ} - \theta) = \frac{a}{b}$$

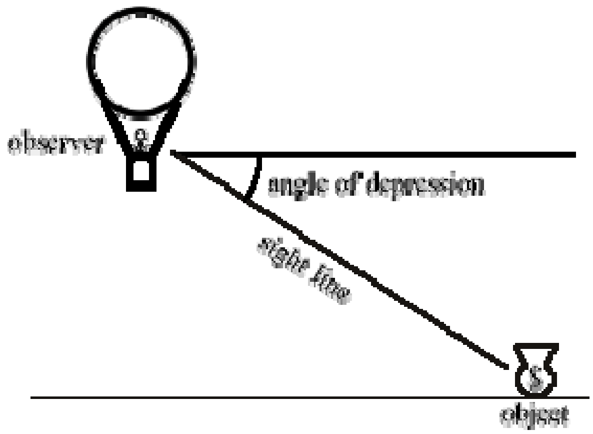


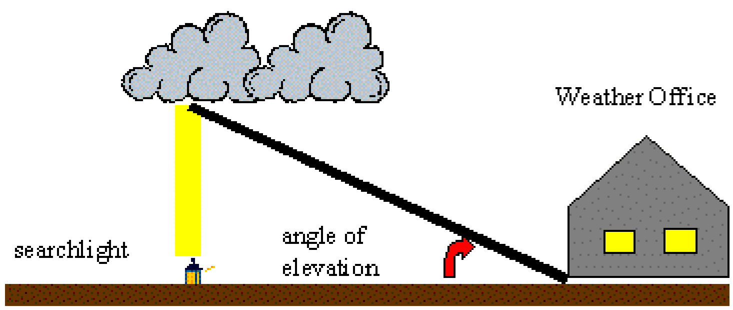
Hence we conclude that

$$\cos(90^{\circ} - \theta) = \sin\theta \;, \quad \sin(90^{\circ} - \theta) = \cos\theta \;, \quad \tan(90^{\circ} - \theta) = \frac{1}{\tan\theta}$$

Applications of Trigo Ratios – Angle of elevation and Angle of depression



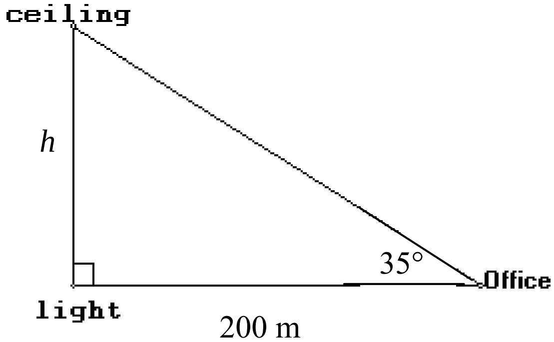




The airport meteorologists keep an eye on the weather to ensure the safety of the flights. One thing they watch is the cloud ceiling. The cloud ceiling is the lowest altitude at which solid cloud is visible. If the cloud ceiling is too low the planes are not allowed to take off or land.

One way a meteorologist can find the cloud ceiling at night is to shine a searchlight that is located a fixed distance from their office vertically into the clouds. Then they measure the angle of elevation to the spot of light on the cloud. The angle of elevation is the angle formed by the line of sight to the spot and the horizontal. Using trigonometry, the cloud ceiling can be determined.

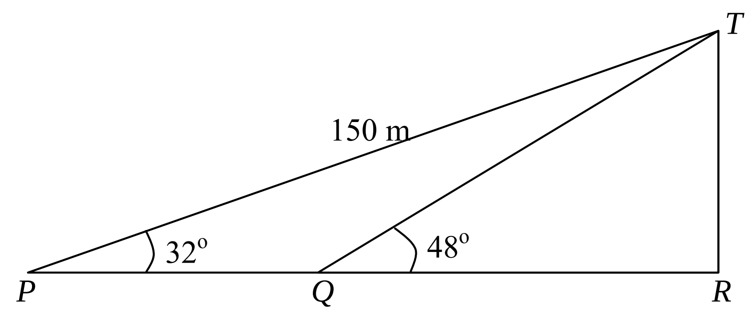
Example 1: A searchlight located 200 meters from a weather office is turned on. If the angle of elevation to the spot of light on the clouds is 35°, how high is the cloud ceiling?



Example 2: A surveyor is 100 meters from the base of a dam. The angle of elevation to the top of the dam measures 26°. The surveyor's eye-level is 1.73 m above the ground. Find the height of the dam.

<u>Exercise 7</u>

Q1 At the point *P*, a boat observes that the angle of elevation of the cliff at point *T* is 32°, and the distance *PT* is 150 m. It sails for a certain distance to reach point *Q*, and observes that the angle of elevation of the point *T* becomes 48°.



(i) Calculate the height of the cliff.

(ii) Calculate the distance the boat is from the cliff at point *Q*.

(iii) Calculate the distance travelled by the boat from point *P* to point *Q*.

Q2 A 3m-ladder is leaning against a vertical wall so that the bottom of the ladder is 1.3 m away from the base of the wall. How large is the angle formed between the ladder and the wall?

Q3 An airplane flying at a speed of 140 m/s begins to climb at an angle of 10°. What is the increase in altitude over the next 15 s?

Q4 At a point 65 m from the base of a cliff, the top of the cliff is seen through an angle of elevation of 37°. How tall is the cliff?

Q5 A tower stands on top of a cliff. At a distance of 60 m from the foot of the cliff which is at ground level, the angles of elevation of the top of the tower as well as the cliff are 65° and 53° respectively. Find the height of the tower.

Q6 A kite is flying at an angle of elevation of 67° from a point on the ground. If 30 m of kite string is out, how far is the kite above the ground?

Q7 Danny sees a balloon that is 30 m above the ground. If the angle of elevation from Danny to the balloon is 75°, how far from Danny is the balloon?

Q8 From a cliff, a person observes a buoy through an angle of depression of 23°. If the cliff is 25 m high, how far is the buoy from the person?

Q9 A man observes the angle of elevation of the top of a mountain to be 50°. He walks 1000 m nearer and finds the angle to be 60°. Find the height of the mountain.

Q10 A ladder of length 18 m leans against the lower edge of a window of a building standing on ground level. It makes an angle of 46° with the horizontal. When it leans against the top edge of the window, it makes an angle of 58° with the horizontal. Find the height of the window.

Q11 A man finds that the angle of elevation of a building is 22°. After walking 20 m towards the building, he finds that the angle of elevation is 33°. Find the height of the building.

Q12 The angle of elevation of the top of a tower from a point B on the ground is 58°. A is a point 35 m further away from the tower than B. From A, the angle of elevation of the top of the tower is 39°. Find the height of the tower.

Q13* Two towers *AD* and *BC* stand vertically on horizontal ground with *C* due east of *D*. Tower *BC* is of height 40 m. An Enemy appears at *E*, where $CE = 60\,\text{m}$, $DC = 100\,\text{m}$, and $\angle DCE = 70°$.

Meanwhile, Sniper Alpha, stationed at *A*, spots the Enemy with an angle of depression of $30°$, Sniper Bravo, stationed at *B*, also spots the enemy.

(a) Calculate the angle of depression of the Enemy from Bravo.

(b) Find the height of tower *AD*.